Physics 513, Quantum Field Theory

Homework 5

Due Tuesday, 7th October 2003

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1. We are to verify the identity

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu.$$

It will be helpful to first have a good representation of $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$. This can be obtained by raising one of the indices of $(\mathcal{J}^{\rho\sigma})_{\lambda\nu}$ which is defined in Peskin and Schroeder's equation 3.18.

$$\begin{aligned}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu = g^{\mu\lambda}(\mathcal{J}^{\rho\sigma})_{\lambda\nu} &= ig^{\mu\lambda}(\delta^\rho_\lambda\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\lambda),\\ &= i(g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu).\end{aligned}$$

We will use this expression for $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$ in the last line of our derivation below. We will proceed by direct computation.

$$\begin{aligned}[\gamma^\mu, S^{\rho\sigma}] &= \frac{i}{4}\left([\gamma^\mu,\gamma^\rho\gamma^\sigma] - [\gamma^\mu,\gamma^\sigma\gamma^\rho]\right),\\ &= \frac{i}{4}\left(\{\gamma^\mu,\gamma^\rho\}\gamma^\sigma - \gamma^\rho\{\gamma^\mu,\gamma^\sigma\} - \{\gamma^\mu,\gamma^\sigma\}\gamma^\rho + \gamma^\sigma\{\gamma^\mu,\gamma^\rho\}\right),\\ &= \frac{i}{2}\left(g^{\mu\rho}\gamma^\sigma - \gamma^\rho g^{\mu\sigma} - g^{\mu\sigma}\gamma^\rho + \gamma^\sigma g^{\mu\rho}\right),\\ &= i\left(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho\right),\\ &= i\left(g^{\mu\rho}\delta^\sigma_\nu\gamma^\nu - g^{\mu\sigma}\delta^\rho_\nu\gamma^\nu\right),\\ &= i\left(g^{\mu\rho}\delta^\sigma_\nu - g^{\mu\sigma}\delta^\rho_\nu\right)\gamma^\nu,\\ \therefore [\gamma^\mu, S^{\rho\sigma}] &= (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu.\end{aligned}$$

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2. All of the required identities will be computed by directly.

a) $\gamma_\mu\gamma^\mu = 4$

$$\gamma_\mu\gamma^\mu = (\gamma^0)^2 + (\gamma^1)^2 + (\gamma^2)^2 + (\gamma^3)^2 = 4.$$

b) $\gamma_\mu \not{k}\gamma^\mu = -2\not{k}$

$$\begin{aligned}\gamma_\mu\not{k}\gamma^\mu &= \gamma_\mu\gamma_\nu k^\nu\gamma^\mu,\\ &= (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)k^\nu\gamma^\mu,\\ &= 2k_\mu\gamma^\mu - \gamma_\nu k^\nu\gamma_\mu\gamma^\mu,\\ \therefore \gamma_\mu\not{k}\gamma^\mu &= -2\not{k}\end{aligned}$$

c) $\gamma_\mu\not{p}\not{q}\gamma^\mu = 4p\cdot q$

$$\begin{aligned}\gamma_\mu\not{p}\not{q}\gamma^\mu &= \gamma_\mu\gamma_\nu p^\nu q_\rho\gamma^\rho\gamma^\mu,\\ &= (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)p^\nu q_\rho(2g^{\rho\mu} - \gamma^{\mu\rho}),\\ &= (2p_\mu - \not{p}\gamma_\mu)(2q^\mu - \not{q}\gamma^\mu),\\ &= 4p\cdot q - 2\not{p}\not{q} - 2\not{p}\not{q} + 4\not{p}\not{q},\\ \therefore \gamma_\mu\not{p}\not{q}\gamma^\mu &= 4p\cdot q.\end{aligned}$$

d) $\gamma_\mu\not{k}\not{p}\not{q}\gamma^\mu = -2\not{p}\not{q}\not{k}$

By repeated use of the identity $\gamma^\mu\gamma^\nu = 2g^{\mu\nu} - \gamma^\nu\gamma^\mu$,

$$\begin{aligned}\gamma_\mu\not{k}\not{p}\not{q}\gamma^\mu &= \gamma_\mu\gamma^\nu k_\nu\gamma^\rho p_\rho\gamma^\sigma q_\sigma\gamma^\mu,\\ &= 2\gamma_\mu\not{k}\not{p}q_\sigma g^{\sigma\mu} - 2\gamma_\mu\not{k}p_\rho g^{\rho\mu}\not{q} + 2\gamma_\mu k_\nu g^{\nu\mu}\not{p}\not{q} - 4\not{k}\not{p}\not{q},\\ &= 2\not{q}\not{k}\not{p} - 2\not{p}\not{k}\not{q} - 2\not{k}\not{p}\not{q},\\ &= 4\not{q}k\cdot p - 2\not{q}\not{p}\not{k} - 4p\cdot k\not{q},\\ \therefore \gamma_\mu\not{k}\not{p}\not{q}\gamma^\mu &= -2\not{p}\not{q}\not{k}.\end{aligned}$$

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3. We are to prove the Gordon identity,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p).$$

Explicitly writing out each term in the brackets and recalling the anticommutation relations of γ^μ, the right hand side becomes,

$$\begin{aligned}\bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) &= \bar{u}(p')\left[\frac{1}{2m}\left(p'^\mu + p^\mu - \tfrac{1}{2}\gamma^\mu\gamma^\nu(p-p')_\nu + \tfrac{1}{2}\gamma^\nu\gamma^\mu(p-p')_\nu\right)\right]u(p),\\ &= \bar{u}(p')\left[\frac{1}{2m}\left(p'^\mu + p^\mu - \tfrac{1}{2}\gamma^\mu\gamma^\nu(p-p')_\nu + g^{\nu\mu}(p-p')_\nu - \tfrac{1}{2}\gamma^\mu\gamma^\nu(p-p')_\nu\right)\right]u(p),\\ &= \bar{u}(p')\left[\frac{1}{2m}\left(2p'^\mu - \gamma^\mu\gamma^\nu(p-p')_\nu\right)\right]u(p),\\ &= \bar{u}(p')\left[\frac{1}{2m}\left(2p'^\mu - \gamma^\mu \not{p} - \gamma^\mu \not{p}'\right)\right]u(p).\end{aligned}$$

Now, recall that the Dirac equation for $u(p)$ is

$$\not{p}u(p) = mu(p).$$

Converting this for $\bar{u}(p')\not{p}'$, one obtains

$$\bar{u}(p')\not{p}' = m\bar{u}(p').$$

Applying both of these equations where we left of, we see that

$$\bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) = \bar{u}(p')\frac{p'^\mu}{m}u(p).$$

Looking again at the Dirac equation, $m\bar{u}(p') = \bar{u}(p')\not{p}' = \bar{u}(p')\gamma^\mu p'_\mu$, it is clear that

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p).$$

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4. **a)** To demonstrate that $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes each of the γ^μ, it will be helpful to remember that whenever $\mu \neq \nu$, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ by the anticommutation relations. Therefore, any odd permutation in the order of some γ's will change the sign of the expression. It should therefore be quite clear that

$$\begin{aligned}\gamma^5\gamma^0 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 = -i\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^5;\\ \gamma^5\gamma^1 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 = i\gamma^0\gamma^2\gamma^3 = -i\gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^5;\\ \gamma^5\gamma^2 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^3 = -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^5;\\ \gamma^5\gamma^3 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^3 = i\gamma^0\gamma^1\gamma^2 = -i\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^3\gamma^5;\\ &\therefore \{\gamma^5, \gamma^\mu\} = 0.\end{aligned}$$

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b) We will first show that γ^5 is hermitian. Note that the derivation relies on the fact that $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. These facts are inherent in our chosen representation of the γ matrices.

$$\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger,\\ &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger,\\ &= i\gamma^3\gamma^2\gamma^1\gamma^0,\\ &= -i\gamma^2\gamma^1\gamma^0\gamma^3,\\ &= -i\gamma^1\gamma^0\gamma^2\gamma^3,\\ &= i\gamma^0\gamma^1\gamma^2\gamma^3,\\ &= \gamma^5.\end{aligned}$$

Let us now show that $(\gamma^5)^2 = 1$.

$$\begin{aligned}(\gamma^5)^2 &= -i\gamma_3\gamma_2\gamma_1\gamma_0 i\gamma^0\gamma^1\gamma^2\gamma^3,\\ &= \gamma_3\gamma_2\gamma_1\gamma_0\gamma^0\gamma^1\gamma^2\gamma^3,\\ &= \gamma_3\gamma_2\gamma_1\gamma^1\gamma^2\gamma^3,\\ &= \gamma_3\gamma_2\gamma^2\gamma^3,\\ &= \gamma_3\gamma^3,\\ &= 1.\end{aligned}$$

c) Note that $\epsilon_{\kappa\lambda\mu\nu}$ is only nonzero when $\kappa \neq \lambda \neq \mu \neq \nu$ which leaves exactly $4! = 24$ nonzero terms from the 24 possible permutations. Also note that $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$, like $\epsilon_{\kappa\lambda\mu\nu}$, is totally antisymmetric–any odd permutation of indices changes the sign of the argument. Therefore, they change sign exactly together, $\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ does not change sign. That is to say that each of the 24 nonzero terms of $\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ is identical to $\epsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3$. So

$$\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = 24\epsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{24}{i}\gamma^5,$$

$$\therefore \gamma^5 = -\frac{i}{24}\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu.$$

This implies that

$$\gamma^5 = -i\epsilon_{\kappa\lambda\mu\nu}\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]},$$

$$\therefore \gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5.$$

5. We will begin by simply directly computing the form of $\xi_\pm$ from the eigenvalue equation

$$(\hat{\mathbf{p}}\cdot\tfrac{1}{2}\vec{\sigma})\,\xi_\pm(\hat{\mathbf{p}}) = \pm\tfrac{1}{2}\xi_\pm(\hat{\mathbf{p}}).$$

Let us begin to expand the left hand side of the eigenvalue equation,

$$(\hat{\mathbf{p}}\cdot\tfrac{1}{2}\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\cos\theta & 0 \\ 0 & -\cos\theta\end{pmatrix},$$

$$\therefore\ (\hat{\mathbf{p}}\cdot\tfrac{1}{2}\vec{\sigma}) = \frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}.$$

Note that we can see here that because this matrix has determinant -1 and trace 0, the eigenvalues must be are ± 1. Therefore, we may write the eigenvalue equation as the system of equations,

$$\frac{1}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\begin{pmatrix}\xi^1_\pm \\ \xi^2_\pm\end{pmatrix} = \pm\frac{1}{2}\begin{pmatrix}\xi^1_\pm \\ \xi^2_\pm\end{pmatrix}.$$

These two equations are equivalent; I will use the first row of equations. This becomes

$$\pm\xi^1_\pm = \cos\theta\xi^1_\pm + \sin\theta e^{-i\phi}\xi^2_\pm.$$

Therefore,

$$\xi^1_+ = \frac{\sin\theta e^{-i\phi}\xi^2_+}{1-\cos\theta} = e^{-i\phi}\tan(\theta/2)\xi^2_+ \qquad \text{and} \qquad \xi^1_- = -\frac{\sin\theta e^{-i\phi}\xi^2_-}{1+\cos\theta} = -e^{-i\phi}\tan(\theta/2)\xi^2_-$$

So that

$$\xi_+ = \begin{pmatrix} e^{-i\phi}\cot(\theta/2)\xi^2_+ \\ \xi^2_+\end{pmatrix} \qquad \text{and} \qquad \xi_- = \begin{pmatrix} -e^{-i\phi}\tan(\theta/2)\xi^2_- \\ \xi^2_-\end{pmatrix}.$$

To find the normalization, we must apply the normalization conditions $\xi^\dagger_\pm\xi_\pm = 1$. By direct calculation,

$$\begin{aligned}\xi^\dagger_+\xi_+ = 1 &= (\xi^2_+)^2(\cot^2(\theta/2)+1),\\ &= \frac{(\xi^2_+)^2}{\sin^2(\theta/2)},\\ \therefore \xi^2_+ &= e^{i\eta^+}\sin(\theta/2).\end{aligned}$$

Likewise for ξ_-,

$$\begin{aligned}\xi_-^\dagger\xi_- = 1 &= (\xi_-^2)^2(\tan^2(\theta/2)+1),\\ &= \frac{(\xi_-^2)^2}{\cos^2(\theta/2)},\\ \therefore \xi_-^2 &= e^{i\eta^-}\cos(\theta/2).\end{aligned}$$

Notice that if ξ_+ satisfies $\xi^\dagger\xi = 1$ then so does $\xi' = e^{i\eta}\xi$. So in solving the normalization equations, we necessarily introduced an arbitrary phase η. Noting, this, spinors become

$$\xi_+ = e^{i\eta^+}\begin{pmatrix} e^{-i\phi}\cos(\theta/2) \\ \sin(\theta/2)\end{pmatrix} \qquad \text{and} \qquad \xi_- = e^{i\eta^-}\begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2)\end{pmatrix}.$$

Lastly, we would like to set the phase η so that when the particle is moving in the $+z-$direction, they reduce to the usual spin-up/spin-down forms. It should be quite obvious that $\eta^- = 0$ satisfies this condition for ξ_-. For ξ^+, we will set the phase to $\eta^+ = \phi$ so that we may lose the $e^{-i\phi}$ term when $\theta = 0$. So we may write our final spinors as

$$\xi_+ = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2)\end{pmatrix} \qquad \text{and} \qquad \xi_- = \begin{pmatrix} -e^{-i\phi}\sin(\theta/2) \\ \cos(\theta/2)\end{pmatrix}.$$

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